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Non-planar double-box, massive and massless pentabox Feynman integrals in the negative-dimensional approach

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Abstract
The negative-dimensional integration method is a technique which can be applied, with success, in usual covariant gauge calculations. We consider three two-loop diagrams: the scalar massless non-planar double-box with six propagators and the scalar pentabox in two cases, where six virtual particles have the same mass, and in the case all of them are massless. Our results are given in terms of hypergeometric functions of Mandelstam variables and also for arbitrary exponents of propagators and dimension D.

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1. Introduction

Studies in particle phenomenology require more and more sophisticated calculations [1], and the measurement of the $(g-2)$ factor has now an error of 1 ppb order of magnitude [2], thanks to the perturbative approach. Within this perspective there has been some interest in massless double-box—planar and non-planar [3]—and pentabox integrals. Smirnov [4] studied the former, using the Mellin–Barnes (MB) technique, in three cases: scalar and tensorial with four legs on-shell, and scalar with one leg off-shell; Tausk [5] and Smirnov and Veretin [7], also using the MB method, presented explicit results for non-planar, or crossed, double-box with seven and six propagators, in the special case where the exponents of propagators are equal to 1; Kümmer *et al* [8] came across with the same integrals studying the potential between quarks in the Coulomb gauge; Anastasiou *et al* [9] calculated the latter, in the integration-by-parts approach, for both the scalar and the tensorial cases with massless internal particles. The results which were obtained with the MB approach are, like the ones obtained by the negative-dimensional integration method (NDIM), expressed in terms of infinite series of hypergeometric type. Of course, progress along this line is greater in covariant gauges;

however, perturbative calculations in non-covariant ones are also carried out and sometimes need more powerful techniques than the former [10].

In this paper we choose to tackle the scalar on-shell non-planar double-box integral for arbitrary exponents of propagators, a result that is missing in the literature and the pentabox integral in two cases: where all particles are massless, and where six virtual particles have the same mass, a diagram which, for instance, contributes to photon–photon scattering, and as far as we know has not yet been calculated. NDIM gives us several results in terms of Mandelstam variables and masses, each valid in a certain kinematical region. We give the results in terms of hypergeometric series for arbitrary exponents of propagators and dimension. In our approach, no reduction formulae or integration-by-parts methods are used or even necessary. It is also worth observing that NDIM is a technique which can be applied to other gauge choices, such as the Coulomb and the light-cone gauges [11].

An important feature of NDIM is that *it is not* a regularization technique. It is worth remembering Dunne and Halliday [12]: the negative-dimensional integrals (in D-dimensions) can be related to positive-dimensional ones (in $2N$-dimensions) over Grassmannian variables; in fact, one has just to make $D \leftrightarrow -2N$. So, in the NDIM context there are no singularities, no poles, etc. However, when we perform the analytic continuation in order to allow negative exponents of propagators and positive dimension, poles appear for specific values of these exponents and physical $D = 4$ dimensions and we have the same results which other techniques provide. This is therefore a consistent method to solve Feynman loop integrals pertaining to the usual covariant or non-covariant algebraic gauges, such as Coulomb and light-cone ones (even at the two-loop level).

The aim of our paper is not to establish the axiomatic foundations for the NDIM nor to demonstrate in a rigorous mathematical sense its principles and basis. Instead, given the simple steps that the methodology requires to work out complicated Feynman integrals, we are interested in testing it to the limits of our present calculational abilites. For this purpose we are presenting here the exact results yielded by this method in another [6] true two-loop calculation. Such results must be compared with those obtained using different techniques so that not only are previous answers double checked and the confidence in the novel method increased, but also in order to demonstrate the feasibility of the latter in performing the calculations.

The outline of our paper is as follows: in section 2 we study pentabox integrals, where in one case virtual particles are massless and in the other, we consider six of the virtual particles massive with equal masses, a graph which contributes to photon–photon scattering (which were considered recently in [13]), although a full calculation of such an effect and an evaluation of the beta functions in physical processes are beyond the scope of the present paper. We also solve the scalar massless non-planar double-box with six propagators. In section 3, we present our conclusions.

2. Pentabox and non-planar double-box integrals

Let us define the three relevant negative-dimensional integrals, namely,

$$\mathcal{P} = \int\!\!\int \mathrm{d}^D q\, \mathrm{d}^D k_1 \mathrm{P}(q, k_1, p, p', p_1) \tag{1}$$

$$\mathcal{MP} = \int\!\!\int \mathrm{d}^D q\, \mathrm{d}^D k_1 \mathrm{MP}(q, k_1, p, p', p_1, \mu) \tag{2}$$

and

$$\mathcal{NP}_6 = \int\!\!\int \mathrm{d}^D q\, \mathrm{d}^D r\, \mathrm{NP}_6(q, r, p_2, p_3, p_4) \tag{3}$$

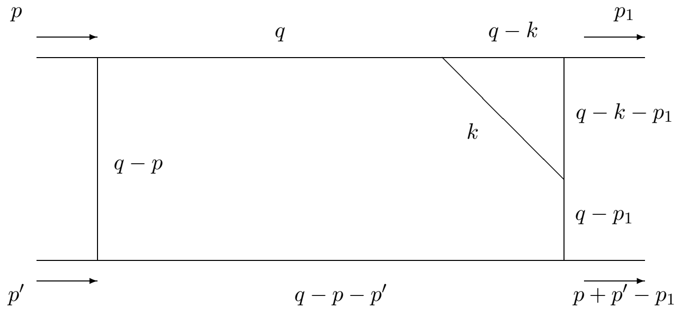

Figure 1. Scalar massless pentabox with all external legs on-shell. Mandelstam variables are defined as $s=(p+p')^2$ and $t=(p-p_1)^2$.

where the integrands are respectively

$$\mathrm{P} \equiv (q^2)^i (q-p)^{2j} (q-k_1)^{2k} (q-k_1-p_1)^{2l} \left(k_1^2\right)^m (q-p_1)^{2n} (q-p-p')^{2r} \tag{4}$$

$$\begin{aligned}\mathrm{MP} \equiv{}& (q^2-\mu^2)^i [(q-p)^2-\mu^2]^j [(q-k_1)^2-\mu^2]^k [(q-k_1-p_1)^2-\mu^2]^l \left(k_1^2\right)^m \\ &\times [(q-p_1)^2-\mu^2]^n [(q-p-p')^2-\mu^2]^r \end{aligned}\tag{5}$$

$$\mathrm{NP}_6 \equiv (q^2)^i (q-p_3)^{2j} (q+r)^{2k} (q+r+p_2)^{2l} (r^2)^m (r-p_4)^{2n} \tag{6}$$

which represent massless pentabox, massive pentabox and non-planar double-box, respectively. Once they are introduced, we evaluate them using the NDIM.

When Halliday and co-workers advanced the idea of negative-dimensional integration, they proved that it is equivalent [12] to Grassmannian integration in the positive dimension, the correspondence being as simple as $D \leftrightarrow -2N$. This fact is implied in the very structure of the above integrands.

2.1. Massless pentabox integral

The negative-dimensional integral for massless pentabox diagram (see figure 1) has the following generating functional:

$$\begin{aligned} G_{\mathcal{P}} ={}& \iint \mathrm{d}^D q \, \mathrm{d}^D r' \exp\big[-\alpha q^2 - \beta (q-p)^2 - \gamma (q-r')^2 - \theta (q-r'-p_1)^2 \\ &- \phi r'^2 - \eta (q-p_1)^2 - \omega (q-p-p')^2\big] \end{aligned}\tag{7}$$

$$= \left(\frac{\pi^2}{\Lambda}\right)^{D/2} \exp\left[-\frac{1}{\Lambda}\left(\alpha\lambda_2\omega s + \beta\eta\lambda_2 t + \beta\theta\phi t\right)\right] \tag{8}$$

where $\lambda_2 = \gamma+\theta+\phi$ and $\Lambda = (\alpha+\beta+\eta+\omega)\lambda_2 + \gamma\phi + \theta\phi$.

Taylor expanding (7), we have our negative-dimensional integral $\mathcal{P}$ as a factor in a seven fold summation series,

$$G_{\mathcal{P}} = \sum_{i,j,k,l,m,n,r=0}^{\infty} \frac{(-1)^{i+j+k+l+m+n+r}}{i!j!k!l!m!n!r!} \alpha^i \beta^j \gamma^k \theta^l \phi^m \eta^n \omega^r \mathcal{P}. \tag{9}$$

On the other hand, taking (8) and making an expansion (including a multinomial one) in power series, we obtain

$$\begin{aligned} G_{\mathcal{P}} ={}& \sum_{X,Y,Z=0}^{\infty} \frac{\alpha^{X_{123}+Y_{123}} \beta^{X_{4567}+Y_{456}} \gamma^{X_{14}+Y_{147}+Z_{13}} \theta^{X_{257}+Y_{258}+Z_{24}} \phi^{X_{367}+Y_{369}+Z_{125}} \eta^{X_{456}+Y_{789}} \omega^{X_{123}+Z_{345}}}{X_1!\cdots X_7! y_1!\cdots Y_9! Z_1!\cdots Z_5!} \\ &\times (-s)^{X_{123}} (-t)^{X_{4567}} (-X_{1234567} - D/2)! \end{aligned}\tag{10}$$

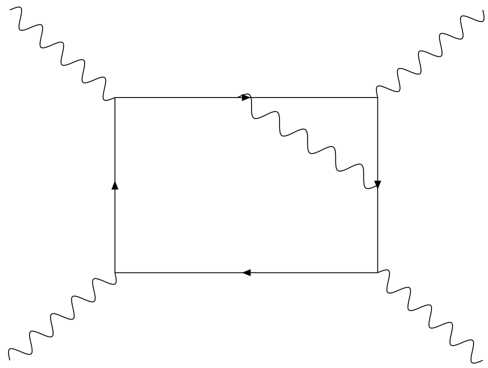

Figure 2. Pentabox diagram with six massive propagators. We consider the case of equal masses and external particles on-shell. At the two-loop level it contributes to photon–photon scattering.

where the sum over X, Y, Z is a shorthand notation for seven-, nine- and five-fold sums, respectively. Moreover, we define $X_{12} = X_1 + X_2, \ldots$, and so forth.

Therefore, the exponential above generates a series indexed by 21 indices, while the seven propagators in the argument of the integrand give rise to seven equations and the multinomial expansion to another one, see equation (9). Now, solving both equations for $\mathcal{P}$ we conclude that there must be some relation among the two sets of indices $\{X, Y, Z\}$ and $\{i, j, k, l, m, n, r\}$. It is a system of algebraic equations

$$\begin{aligned} &X_{123} + Y_{123} = i \qquad X_{4567} + Y_{456} = j \qquad X_{14} + Y_{147} + Z_{13} = k \\ &X_{257} + Y_{258} + Z_{24} = l \qquad X_{367} + Y_{369} + Z_{125} = m \qquad X_{456} + Y_{789} + Z_{13} = n \\ &X_{123} + Z_{345} = r \qquad \sum X + \sum Y + \sum Z = -D/2 \end{aligned} \tag{11}$$

for which there is no unique solution, since we have 21 'unknowns' and just eight equations. In fact, there are 203 490 possible 8×8 systems which can be solved in terms of exponents of propagators i, j, k, l, m, n, r, dimension D and some of X, Y, Z. The computer can calculate such 8×8 determinants easily: 134 890 of them are zero, i.e, they give empty sets of solutions.

So, our result will be written in terms of a 13-fold series of hypergeometric type. This can be worked out conveniently and simplified, since we have three possible variables: t/s, s/t and unity. Series which depend on both t/s and s/t cannot be convergent, so we will not consider them.

Next, our strategy is to search for the simplest hypergeometric series among the remaining 68 600 solutions. The criterion for this search is dictated by the fact that the more sums with unit argument, the simpler it is, since one can sum them (at least in principle, provided some relations amongst the parameters are observed) using Gauss' summation formula [14],

$${}_2F_1(a, b; c|1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \qquad \mathrm{Re}(c-a-b) > 0. \tag{12}$$

In other words, when some of these 13 sums can be rewritten as gamma functions. Of course, not all sums with unit argument can be summed this way, as we will shortly see. These cases occur when the resulting function is of the type ${}_3F_2(\ldots|1)$, or even more complex.

There are many different ways in which the 13-fold series appear, and we can classify them according to the following general form:

$$(\text{Momenta})(\text{Gammas}) \sum_{\text{all}=0}^{\infty} \frac{z^A}{n_1! \cdots n_9! m_1! \cdots m_4!}$$

where z is one of the three possible variables t/s, s/t or 1, and A represents $n_{123} = n_1 + n_2 + n_3$, $n_{1234} = \sum_{i=1}^{4} n_i$ or $n_{12345} = \sum_{i=1}^{5} n_i$.

The simplest hypergeometric series representation for the scalar massless pentabox is given by hypergeometric series with three such 'variables' (here we use the word 'variables', to denote the variable s/t appearing thrice as the summation variable within the series),

$$\mathcal{P}_3 = \pi^D s^{\sigma-j} t^j P_3^{AC} \sum(Y_4, Y_5, Y_6)\, {}_3F_2(a_3, b_3, c_3; e_3, f_3|1) \tag{13}$$

where

$$\begin{aligned}\sum(Y_4, Y_5, Y_6) \\ \equiv \sum_{Y_4,Y_5,Y_6=0}^{\infty} & \frac{(-k|Y_4)(j-k-l-m-n-D/2|-Y_{456})(D+k+l+m+n+r|Y_{456})}{(1-k-m-D/2|-Y_5)(D+k+l+m|Y_{456})} \\ & \times \frac{(D+i+k+l+m+n|Y_{456})(k+l+D/2|Y_6)(-j|Y_{456})}{(1+\sigma-j|Y_{456})} \frac{(s/t)^{Y_{456}}}{Y_4!\,Y_5!\,Y_6!}\end{aligned} \tag{14}$$

and the parameters of ${}_3F_2$ are

$$\begin{aligned} &a_3 = -k + Y_4 \qquad b_3 = j-k-l-m-n-D/2-Y_{456} \\ &c_3 = -k-l-m-D/2 \qquad e_3 = 1-k-m-D/2-Y_5 \\ &f_3 = -k-l-m-n-D/2 \end{aligned} \tag{15}$$

with $\sigma = i+j+k+l+m+n+r+D$ and

$$\begin{aligned} P_3^{AC} = {} & (-i|\sigma-j)(-k-l-m-n-D/2|j)(-l|k+l+m+D/2)(-m|-k-l-D/2) \\ & \times (-r|k+l+r+D/2)(\sigma+D/2|-2\sigma-D/2+j)(k+l+m+D|i+n). \end{aligned} \tag{16}$$

In the above equations, we have used the subscript '3' and the superscript 'AC' to mean that we have three 'variables' and the result is analytically continued into positive dimension D. Pochhammer symbols and some of their properties are used throughout the expressions,

$$(a|b) \equiv (a)_b = \frac{\Gamma(a+b)}{\Gamma(a)} \qquad (a|-k) = \frac{(-1)^k}{(1-a|k)} \qquad (a|b+c) = (a+b|c)(a|b). \tag{17}$$

Observe that in (13) there is a fourth sum, namely in Z_1, which has unit argument and could not be summed because it is a ${}_3F_2$ hypergeometric series which can be expressed in terms of gamma functions only in some special cases. This means that we were able to sum up to nine series (with unit arguments) using (12).

The second type of result provided by our method is given by hypergeometric series with four 'variables',

$$\begin{aligned}\mathcal{P}_4 = \pi^D s^\sigma P_4^{AC} \sum_{X_n=0}^{\infty} & \frac{(-j|X_{4567})(-n|X_{456})(-k|X_4)(-l|X_{57})(-\sigma|X_{4567})}{(i-\sigma|X_{4567})(r-\sigma|X_{4567})(-k-l|X_{457})} \\ & \times \frac{(D/2+m|X_{45})(D/2+k+l|X_6)(-k-l-m-D/2|X_7)}{(D+k+l+m|X_{456})} \frac{(t/s)^{X_{4567}}}{X_4!X_5!X_6!X_7!}\end{aligned} \tag{18}$$

where the subscript '4' means a four-fold hypergeometric series representation for $\mathcal{P}$, and

$$\begin{aligned} P_4^{AC} = {} & (-i|\sigma)(-r|\sigma)(-k-l|-m-D/2)(\sigma+D/2|-2\sigma-D/2) \\ & \times (k+l+m+D|-m-D/2)(-m|2m+D/2). \end{aligned} \tag{19}$$

Note that this fourfold series can be used to study forward scattering. Taking $t = 0$ we are left with only the first term in the series, which is equal to unity, that is, the series collapses and the result is merely (the superscript FS for the forward scattering case)

$$\mathcal{P}_4^{[FS]} = \pi^D s^\sigma P_4^{AC}.$$

The last one, given by hypergeometric series with five 'variables',

$$\mathcal{P}_5 = \pi^D s^{i+j+r+D/2} t^{k+l+m+n+D/2} P_5^{AC} \sum_{Y_n, Z_m=0}^{\infty} \\ \times \frac{(-n|Y_{789})(-k|Y_7+Z_1)(-k-l-m-D/2|Z_{12})(k+l+D/2|Y_9)}{(1-k-m-D/2|Y_{789})(1+j-k-l-m-n-D/2|Y_{789}+Z_{12})(D+k+l+m|Y_{789})} \\ \times \frac{(j+r+D/2|Y_{789}+Z_{12})(i+j+D/2|Y_{789}+Z_{12})}{(1+i+j+r+D/2|Y_{789}+Z_{12})} \frac{(s/t)^{Y_{789}+Z_{12}}}{Y_7!Y_8!Y_9!Z_1!Z_2!} \tag{20}$$

where

$$P_5^{AC} = (-i|-j-r-D/2)(-j|k+l+m+n+D/2)(k+l+m+D|-m-D/2) \\ \times (-l|-k-m-D/2)(-m|i+j+m+D/2)(-r|k+m+r+D/2) \\ \times (\sigma+D/2|j+r-\sigma). \tag{21}$$

Observe that in the last line of (20), when we take either $i = -1$ or $r = -1$, those pertinent Pochhammer symbols within the series cancel out, simplifying it.

Of course, there are results which depend on the inverse of such variables, i.e., $\mathcal{P}_3(t/s)$, $\mathcal{P}_4(s/t)$ and $\mathcal{P}_5(t/s)$, which means an interchange of one or more pairs of exponents of propagators and $s \leftrightarrow t$.

How are these different expressions related to one another? How do they relate to previous calculations by other methods, for example for special cases of the propagator powers? How unique or ambiguous is the analytic continuation in dimension to get back to answers in positive dimensions? What form of renormalization is necessary to connect with conventional calculations?

To answer the first question one must recall that if two series (for instance $\mathcal{P}_3$ and $\mathcal{P}_5$) represent the same function (the integral $\mathcal{P}$), they must be related by analytic continuation [15]. So, the results provided by the NDIM are always related by analytic continuation either directly (overlapping regions of convergence) or indirectly (no overlapping of regions). In the present case there are no previous calculations in the literature considering arbitrary exponents of propagators.

2.2. Massive pentabox integral

Introducing masses in the NDIM context is very simple [16]. The generating functional becomes

$$G_{MP} = \iint \mathrm{d}^D q \,\mathrm{d}^D k_1 \exp\{-\alpha(q^2-\mu^2) - \beta[(q-p)^2-\mu^2] - \gamma\left[(q-k_1)^2-\mu^2\right] \\ - \theta\left[(q-k_1-p_1)^2-\mu^2\right] - \phi k_1^2 - \eta\left[(q-p_1)^2-\mu^2\right] \\ - \omega[(q-p-p')^2-\mu^2]\} \tag{22}$$

$$= \left(\frac{\pi^2}{\Lambda}\right)^{D/2} \exp\left[-\frac{1}{\Lambda}(\alpha\lambda_2\omega s + \beta\eta\lambda_2 t + \beta\theta\phi t)\right] \exp[(\alpha+\beta+\gamma+\theta+\eta+\omega)\,\mu^2] \tag{23}$$

Table 1. Column 'terms' indicate where we truncated all the series. Observe that even for a few terms in the series that were summed, we get a good precision. The result is of the form $A + B\epsilon + C\epsilon^2$, where A, B, C are given in the table.

Terms	ϵ^0	ϵ^1	ϵ^2
0	0.167 027 110	−0.012 614 420	0.391 4896
1	0.167 553 265	−0.012 917 188	0.392 7286
2	0.167 558 252	−0.012 920 469	0.392 7776
3	0.167 558 333	−0.012 920 522	0.392 7779
4	0.167 558 335	−0.012 920 523	0.392 7778

so that we have six additional sums, which are generated by the second exponential in the second line above, which corresponds to the massive sector. Besides, the resulting hypergeometric series will have variables of the form

$$\frac{t}{s}, \quad \frac{t}{\mu^2}, \quad \frac{s}{\mu^2}$$

their inverses or unity. Powers of them also occur, such as $\sqrt{s/t}$ and $(s/t)^2$.

Once more we look for convergent series. Among the very large number of possible systems—altogether $27!/(8!19!) = 2220\,075$—there are 1093 289 which have no solution, and the remaining 1126 786 among which we are able to find solutions. So we are left with 49.24% of the total, from which we hunt for the most convenient ones.

The simplest hypergeometric series representation is given by a seven-fold summation,

$$\begin{aligned}(\mathcal{MP})_7 = \pi^D (\mu^2)^\sigma P_7^{AC} \sum_{\text{all}=0}^{\infty} &\Big\{[(-i|X_{123})(-j|X_{4567})(-k|X_{14})(-l|X_{257})(-m|X_{367}) \\ &\times (-n|X_{456})(-r|X_{123})(-\sigma|X_{1234567})]\big/[(-m-\sigma|X_{1245}+2X_{367}) \\ &\times (-k-l|X_{12457})(D/2|X_{1234567})]\Big\} \\ &\times \Big\{[(-i-j-m-n-r-D/2|X_{12457}+2X_{36})(-k-l-m-D/2|X_7) \\ &\times (D/2+m|X_{1245})]\big/[(-i-j-n-r|2X_{123456}+X_7)X_1!\cdots X_7!]\Big\} \\ &\times \left(\frac{s}{\mu^2}\right)^{X_{123}} \left(\frac{t}{\mu^2}\right)^{X_{4567}} \end{aligned} \tag{24}$$

where

$$P_7^{AC} = (D/2|m)(-\sigma|-m)(-i-j-n-r|-m-D/2)(-k-l|-m-D/2). \tag{25}$$

As a sample numerical calculation we give in table 1 an expansion in the $\epsilon = 2 - D/2$ parameter.

Hypergeometric series with ten summation indices, nine 'variables', also occur,

$$\begin{aligned}(\mathcal{MP})_9 = \pi^D t^\sigma P_9^{AC} \sum_{\text{all}=0}^{\infty} &\Big\{[(-i|X_{123}+W_1)(-k|W_3+X_1+Z_1)(-n|W_5) \\ &\times (-r|W_6+X_{123})(-\sigma|W_{123456}+X_{123})]\big/(1+j-\sigma|X_{123}+W_{13456})\Big\} \\ &\times \Big\{[(1-k-l-m-D|W_{34}-X_{123})(k+m+D/2|X_2-W_3-Z_1) \\ &\times (-k-l-m-D/2|W_{34}+Z_1)]\big/(1-k-l-D/2|W_{34}-X_3)\Big\} \\ &\times \frac{(i+j+r+D/2|-W_{126}-X_{123}+Z_1)(1-\sigma-D/2|W_{123456})}{(-k-l-m-n-D/2|W_{345}+Z_1)} \\ &\times \frac{(s/t)^{X_{123}}\,(\mu^2/t)^{W_{123456}}}{X_1!\cdots X_3!W_1!\cdots W_6!Z_1!} \end{aligned} \tag{26}$$

where

$$P_9^{AC} = (-k-l-m-n-D/2|n)(\sigma+D/2|-2\sigma-D/2)(k+l+m+D|-m-D/2) \times(-j|\sigma)(-l|k+l+m+D/2)(-m|i+j+m+r+D/2). \quad (27)$$

Observe that there is a tenth series with unit argument, which is not summable since it is a ${}_3F_2(a,b,c;e,f|1)$. Finally, the last result we present for the massive pentabox is a 11-fold series,

$$(\mathcal{MP})_{11} = \pi^D(\mu^2)^{i+j+r+D/2}t^{k+l+m+n+D/2}P_{11}^{AC}\times\sum_{\text{all}=0}^{\infty}\big\{[(-i|X_{123})(-k|X_1+W_3+Y_7+Z_1) \times(-n|W_5+Y_{789})(-r|X_{123})(k+l+D/2|X_3+Y_9-W_{34})]\big/ (1+j-k-l-m-n-D/2|W_{345}+Y_{789}+Z_{12})\big\} \times\big\{[(1-k-l-m-n-D|-X_{123}+Y_{789}+Z_{12}+W_{345}) \times(1-k-l-m-D|W_{34}-X_{123}-Y_{789})]\big/ [(1-k-l-m-n-D|W_{345}-X_{123}) \times(1-k-m-D/2|W_3-X_2-Y_8+Z_1)]\big\} \times\frac{(-i-j-r-D/2|X_{123}-Y_{789}-Z_{12})(-k-l-m-D/2|W_{34}+Z_{12})}{(-i-j+k+l+m+n-r+D/2|-W_{345}+2X_{123}-Y_{789}-Z_{12})} \times\frac{(s/\mu^2)^{X_{123}}(\mu^2/t)^{W_{345}+Y_{789}+Z_{12}}}{X_1!X_2!X_3!Y_7!Y_8!Y_9!Z_1!Z_2!W_3!W_4!W_5!} \quad (28)$$

where

$$P_{11}^{AC} = (-i-j+k+l+m+n-r+D/2|i+r)(k+l+m+D|-m-D/2) \times(-j|-i-r-D/2)(-l|-k-m-D/2)(-m|k+2m+D/2). \quad (29)$$

It is important to note that all the series are valid within their regions of convergence [14], e.g., in the first series (24) one must have $|s/\mu^2|<1$ and $|t/\mu^2|<1$, and so on for others.

We mention that there are also other hypergeometric series, for instance,

$$(\mathcal{MP})_7 = (\mu^2)^{\sigma-n}t^n\sum_{\text{all}=0}^{\infty}\left(\frac{s}{\mu^2}\right)^{X_{abc}}\left(\frac{\mu^2}{t}\right)^{X_{efgh}}\{(\cdots|\cdots)\}_1+(\mu^2)^{\sigma-j}t^j \times\sum_{\text{all}=0}^{\infty}\left(\frac{s}{\mu^2}\right)^{X_{abc}}\left(\frac{\mu^2}{t}\right)^{X_{efgh}}\{(\cdots|\cdots)\}_2 \quad (30)$$

and also 10-fold series, with eight variables, which has poles in the exponents of propagators, such as $\Gamma(i-r)$,

$$(\mathcal{MP})_8 = \Gamma(i-r)\Gamma(\ldots)\sum_{\text{all}=0}^{\infty}\left(\frac{t}{\mu^2}\right)^{X_{abcd}}\left(\frac{\mu^2}{s}\right)^{X_{efgh}} +\Gamma(r-i)\Gamma(\ldots)\sum_{\text{all}=0}^{\infty}\left(\frac{t}{\mu^2}\right)^{X_{abcd}}\left(\frac{\mu^2}{s}\right)^{X_{efgh}} \quad (31)$$

which is regularized introducing [17, 18] for example, $i=-1+\delta$ and then expanding around $\delta=0$.

Among all the possible (more than a million) hypergeometric series representations for the integral in question, there must be many relations among them by analytic continuation (directly or indirectly). Yet, the study of all of them is such a formidable task that for practical purposes it is virtually impossible to even think of doing it thoroughly, so we can only conjecture that the entire set of hypergeometric series representations cover all the kinematical manifold. Also, these analytic continuation formulae can in principle be obtained from them.

It is worth remembering some important points in the complex analysis theory [15]. The reader can observe that some of the results have poles of different order than others, i.e., some of them are like $1/\epsilon^a$ and other(s) are like $1/\epsilon^b$, with $a \neq b$. In our previous work on box integrals for photon–photon scattering, the same occurred [18]. In that case we had a branch cut at $s = 4m^2$, and some hypergeometric functions had a region of convergence which allowed us to study such integrals in that point. On the other hand, such hypergeometric series had a direct analytic continuation into another one which did not allow us to consider $s = 4m^2$. The answer is in [16]: when we carry out an analytic continuation and in the process cross a branch cut, this analytic continuation is not unique and poles do appear. In that case, there were simple and double poles, and we had two possible cases: $F_3 \to \sum F_2$ and $F_3 \to H_2$. In our present case, $(\mathcal{MP})_{11}$ and $(\mathcal{MP})_9$ have third-order poles, and $(\mathcal{MP})_7$ is finite. It depends on the kinematical region we wish to study the pertinent integral.

Back in the 1950s and 1960s [19], a lot of research was done in order to study singularities of Feynman integrals. One of the results we borrow from them is that a graph like the one considered in the present section cannot have singularities in the physical sheet. This is a well known theorem due to Eden (1952). So, in a full calculation of photon–photon scattering the poles must cancel out.

In order to extract specific pole structures of these integrals, we can proceed just like in our previous works on the subject [17]. Expand gamma functions around $\epsilon = 0$ and use Taylor expansion in the hypergeometric series, so that we are left with parametric derivatives of hypergeometric functions. The reader can see detailed calculations in the above referred paper, also in the appendix of [20] and in section 2.3.

2.3. Non-planar double-box integral

Now we turn to massless non-planar double-box with six propagators, namely,

$$\mathcal{NP}_6 = \int \mathrm{d}^D q \mathrm{d}^D r (q^2)^i (q - p_3)^{2j} (q + r)^{2k} (q + r + p_2)^{2l} (r^2)^m (r - p_4)^{2n} \tag{32}$$

which represents the diagram of figure 3. This diagram was also studied by Smirnov and Veretin [7], who presented an explicit result in the case where all exponents were equal to -1. On the other hand, we will write down results for arbitrary values of them.

The generating functional

$$G_{\mathcal{NP}_6} = \int \mathrm{d}^D q \mathrm{d}^D r \exp\{-\alpha q^2 - \beta (q - p_3)^2 - \gamma (q + r)^2 - \theta (q + r + p_2)^2 - \phi r^2 - \omega (r - p_4)^2\} = \left(\frac{\pi^2}{\zeta}\right)^{D/2} \exp\left[-\frac{1}{\zeta}(\beta\gamma\omega s + \alpha\theta\omega t + \beta\theta\phi u)\right] \tag{33}$$

can be integrated without difficulty. We define $\lambda_3 = \alpha + \beta + \gamma + \theta$ and $\zeta = \alpha\gamma + \alpha\theta + \beta\gamma + \beta\theta + \lambda_3(\phi + \omega)$. It is easy to see that Taylor expanding the above exponential will give us three series, while multinomial expansion for ζ will give 12 other series. The equations that form the system to be solved come from the propagators, six altogether, and an additional constraint originates from the multinomial expansion.

So, at the end of the day we are left with $(15-7) = 8$-fold series. Their variables are either

$$\frac{t}{s}, \quad \frac{s}{t}, \quad \frac{t}{u}, \quad \frac{u}{t}, \quad \frac{u}{s}, \quad \frac{s}{u} \tag{34}$$

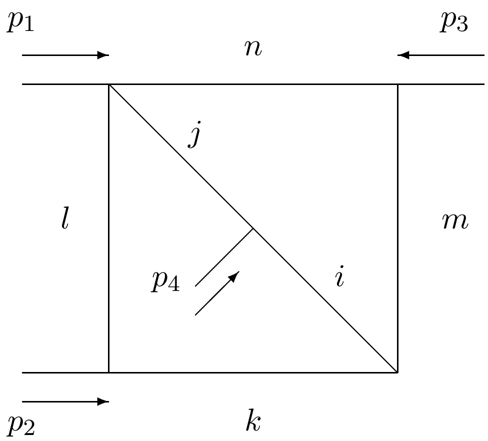

Figure 3. Scalar massless non-planar double-box with six propagators. The labels in the internal lines represent the exponents of propagators, see (3).

and/or unity. The simplest hypergeometric series representation for $\mathcal{NP}_6$ is double series,

$$\mathcal{NP}_6 = \pi^D s^{\sigma'} \Gamma^{AC}_{\rm NP} \sum_{X_2,X_3=0}^{\infty} \frac{(-i|X_2)(-m|X_3)(-\sigma'|X_{23})(-l|X_{23})}{(1+k-\sigma'|X_{23})(1+n-\sigma'|X_3)(1+j-\sigma'|X_2)} \times \frac{(-t/s)^{X_2}\,(u/s)^{X_3}}{X_2!\,X_3!} \tag{35}$$

where $\sigma' = i+j+k+l+m+n+D$ is the sum of exponents and dimension and

$$\Gamma^{AC}_{\rm NP} = (-j|\sigma')(-k|\sigma')(-n|\sigma')(\sigma'+D/2|-2\sigma'-D/2) \times (i+j+m+n+D|-i-j-D/2)(k+l+m+n+D|-m-n-D/2) \times (i+j+k+l+D|-k-l-D/2). \tag{36}$$

The above hypergeometric series reduces to an Appel F_2 function in the special case where $k=-1$,

$$\mathcal{NP}_6 = \pi^D s^{\sigma'} \Gamma^{AC}_{\rm NP}(k=-1) F_2\left(\begin{matrix} -l, -i, -m \\ 1+j-\sigma', 1+n-\sigma' \end{matrix} \middle| \frac{-t}{s}, \frac{u}{s} \right). \tag{37}$$

If we take all the exponents to be equal to -1, we have

$$\mathcal{NP}_6 = \pi^D s^{D-6} \frac{\Gamma(6-D)\Gamma^3(D/2-2)\Gamma^3(D-5)}{\Gamma(3D/2-6)\Gamma^3(D-4)} \times \sum_{X_2,X_3=0}^{\infty} \frac{(1|X_2)(1|X_3)(1|X_{23})}{(6-D|X_3)(6-D|X_2)} \frac{(-t/s)^{X_2}\,(u/s)^{X_3}}{X_2!\,X_3!} \tag{38}$$

$$= \pi^D s^{D-6} \frac{\Gamma(6-D)\Gamma^3(D/2-2)\Gamma^3(D-5)}{\Gamma(3D/2-6)\Gamma^3(D-4)} F_2\left(\begin{matrix} 1,1,1 \\ 6-D, 6-D \end{matrix} \middle| \frac{-t}{s}, \frac{u}{s} \right). \tag{39}$$

This series is the Appel [15] F_2 hypergeometric function which converges when $|t/s| < 1$, $|u/s| < 1$ and $|t/s| + |u/s| < 1$. Note that the above result has a double pole, $1/\epsilon^2$, just as in the works of Tausk [5] and Smirnov and Veretin [7].

One could now ask: would the divergent pieces come from a few terms in the series or from all of them? Clearly, in the present case, divergent factors which generate double and simple poles come from the gamma functions. To write down these poles explicitly, we have to Taylor expand also the hypergeometric function F_2, then we have around $\epsilon = 0$ ($D = 4-2\epsilon$),

$$\begin{aligned} \mathcal{NP}_6 &= \pi^D s^{D-6} \frac{\Gamma(6-D)\Gamma^3(D/2-2)\Gamma^3(D-5)}{\Gamma(3D/2-6)\Gamma^3(D-4)} F_2 \left(\begin{matrix} 1,1,1 \\ 6-D, 6-D \end{matrix} \middle| -\frac{t}{s}, \frac{u}{s} \right) \\ &= \pi^D T s^{D-6} \left[-\frac{3}{\epsilon^2} + 12 + \mathcal{O}(\epsilon) \right] \left\{ F_2 \left(\begin{matrix} 1,1,1 \\ 2,2 \end{matrix} \middle| -\frac{t}{s}, \frac{u}{s} \right) \right. \\ &\quad \left. + 2\epsilon(\partial_\gamma + \partial_{\gamma'}) F_2 + 2\epsilon^2 \left[\partial_\gamma^2 + \partial_{\gamma'}^2 + 4\partial_\gamma \partial_{\gamma'} \right] F_2 + \mathcal{O}(\epsilon^3) \right\} \end{aligned} \tag{40}$$

where γ_E is Euler's constant and

$$\partial_\gamma F_2 = \frac{\partial}{\partial \gamma} F_2 \left(\begin{matrix} \alpha, \beta, \beta' \\ \gamma, \gamma' \end{matrix} \middle| -\frac{t}{s}, \frac{u}{s} \right) \bigg|_{\alpha=\beta=\beta'=1,\ \gamma=\gamma'=2}$$

are called parametric derivatives of hypergeometric functions and can be calculated using Davydychev's [17] approach. (T is given below.)

In order to rewrite the above result as polylogarithm functions, such as Li_2, Li_3 and the usual logarithms, we have to use an integral representation for F_2,

$$\begin{aligned} &\frac{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta)\Gamma(\gamma'-\beta')}{\Gamma(\gamma)\Gamma(\gamma')} F_2 \left[\begin{matrix} \alpha; \beta, \beta' \\ \gamma, \gamma' \end{matrix} \middle| P_1, P_2 \right] \\ &\quad = \int_0^1 \mathrm{d}x_1 \mathrm{d}x_2 \frac{x_1^{\beta-1} x_2^{\beta'-1} (1-x_1)^{\gamma-\beta-1} (1-x_2)^{\gamma'-\beta'-1}}{(1 - x_1 P_1 - x_2 P_2)^\alpha} \end{aligned} \tag{41}$$

where $x_1 + x_2 = 1$. This task is not an easy one at all, since the second derivatives give a very cumbersome result and direct comparison between our result and Tausk's [5] is not possible (we were not able to show that both results are equivalent analytically).

However, the important special case of forward scattering cannot be read off directly from Tausk's result. In contrast, making $t = 0$ or $u = 0$ is extremely simple in our equation (39). Let $u = 0$, then one of the series (in X_3 index) does not contribute and

$$F_2 \left(\begin{matrix} 1,1,1 \\ 6-D, 6-D \end{matrix} \middle| -\frac{t}{s}, 0 \right) = {}_2F_1 \left(\begin{matrix} 1,1 \\ 6-D \end{matrix} \middle| -\frac{t}{s}, 0 \right) = \frac{\Gamma(6-D)\Gamma(4-D)}{\Gamma^2(5-D)} \tag{42}$$

since $s + t + u = 0$ we were able to sum up the series ${}_2F_1$, so $\partial/\partial\gamma$ can be rewritten in terms of $\partial/\partial\epsilon$ and parametric derivatives are easily calculated from equation (42). We quote the result up to order ϵ^0,

$$\begin{aligned} \mathcal{NP}_6|_{u=0} &= \pi^D s^{-2-2\epsilon} T \left\{ -\frac{48}{\epsilon^4} + \frac{201}{2\epsilon^3} - \frac{204}{\epsilon^2} + \frac{8}{\epsilon} \left[83 + 16\psi''(1) - 16\psi''(2) \right] \right. \\ &\quad \left. + 4\left[-134 + 35\psi''(1) - 35\psi''(2) \right] + \mathcal{O}(\epsilon) \right\} \end{aligned} \tag{43}$$

where $\psi(z)\Gamma(z) = \Gamma'(z)$, $\psi'(z)$, $\psi''(z)$ their derivatives and

$$T = \frac{\Gamma(5-D)\Gamma^3(D/2-1)}{\Gamma(3D/2-5)(5-D)}$$

which in fact can be calculated to arbitrary order, since we start from an exact result.

Observe that the result has poles of higher orders, namely ϵ^{-4} and ϵ^{-3}, which come from the fact that we have taken $u = 0$, which also means that the result has a branch point in $u = 0$, a well known fact from the theory of hypergeometric functions.

Of course, the 7×15 system of linear algebraic equations defined by this integral, the 7×7 solvable subsystems are divided into the following categories: among the grand total of 6435 possible solutions, 3519 have no solution at all. Among the remaining 2916 relevant solutions, the NDIM provides other kinds of series, such as 7-fold series and 5-fold series. And all of them have symmetries among *s, t* and *u*, namely,

$$(p_3 \leftrightarrow p_4, j \leftrightarrow n, i \leftrightarrow m, t \leftrightarrow u) \qquad (p_2 \leftrightarrow p_3, l \leftrightarrow n, k \leftrightarrow m, t \leftrightarrow s)$$
$$(p_2 \leftrightarrow p_4, j \leftrightarrow l, i \leftrightarrow k, s \leftrightarrow u). \tag{44}$$

So, for each hypergeometric series representation provided by the NDIM there are another two, also originating from the system of algebraic equations, which represent the same integral and can be transformed into the first using (44). This is the case of (35), and it is stated also by Tausk [5], i.e., the diagram is completely symmetric (through analytic continuation) under external leg permutations.

Despite the complicated form of the non-planar double-box with six propagators, the result we obtained is very simple, a double hypergeometric series, which in the special case of $k = -1$ reduces to the F_2 Appel function. Tausk presented a result for the same graph in terms of di- and trilogarithms, in the special case where all exponents are equal to -1. If we recall [17], Davydychev calculated a scalar integral for photon–photon scattering and transformed the four F_2 Appel hypergeometric functions into dilogarithms. The same occurs here: on one hand di- and trilogarithms and on the other double hypergeometric series, namely F_2.

3. Conclusion

In this paper we considered covariant gauge scalar pentabox and non-planar double-box integrals. For the former we considered two cases: one, all virtual particles being massless, and the other, six of them having the same mass μ while the seventh is massless. The latter was calculated in the massless case and arbitrary exponents of propagators, a result which was missing in the literature. No reduction formulae, i.e. rules to connect a given diagram with simpler ones, were used. The results are given in terms of hypergeometric series and in different kinematical regions.

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